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Oscillations of the Velocity of Sound in Metals in a Magnetic Field*

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A study is made of the effect of a constant magnetic field on the propagation of acoustic waves in metals. It is shown that the velocity of sound may experience two types of oscillations as a function of the intensity of the magnetic field. There exist geometric oscillations associated with the coincidences of the diameter of the cyclotron orbits of the electrons with a half-integral or with an integral multiple of the acoustic wavelength. There are also quantum oscillations which have the same origin as the de Haas–van Alphen effect.

I. INTRODUCTION

IN a previous paper,[1] the author has given a discussion of the dependence of the velocity of acoustic waves on an applied magnetic field. The analysis given in (I) was concerned primarily with the interpretation of the results of the experimental work of Alers and Fleury,[2] and included a relatively detailed study of the effect of a magnetic field on the velocity of acoustic waves in the low-field limit (i.e., when $\omega_c\tau \ll 1$). Further, the assumption was made that $\omega\tau \ll 1$, a condition that is commonly satisfied in actual experiments. However, (I) also contains general expressions valid under a wide variety of conditions. In particular, it is shown there that when $ql \gg 1$ and $\omega_c\tau \gg 1$, the velocity of sound exhibits an oscillatory behavior related to coincidences of the diameter of the cyclotron orbits of the conduction electrons, with a half-integral or with an integral multiple of the acoustic wavelength[3,4] (geometric resonances). The purpose of this paper is to extend the discussion in (I) to effects that occur at low temperatures and high magnetic fields. The notation and the model used in this work are the same as that in (I) so that we shall not, as a rule, define any symbols which have already been defined there. As the title of the work indicates, we shall deal mainly with phenomena involving oscillations in the velocity of sound as a function of the applied magnetic field.

Section II contains a discussion of the quantum effects in the variation of the velocity of sound as a function of the applied magnetic field. We assume first that the conduction electrons within the metal behave as if they were free and then we consider the effect of a more general band structure (we confine our discussion to the independent particle model, however). The result is that the velocity of sound at low temperatures and at high magnetic fields is an oscillatory function of the magnetic field. The character and origin of the oscillations are the same as those occurring in the de Haas–van Alphen effect and, as we shall see in the next section, the effect should be most prominent for longitudinal waves. In Sec. III we use the free electron model to give a discussion of the oscillations in the velocity of sound arising from geometric resonances.

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[1] S. Rodriguez, Phys. Rev. **130**, 1778 (1963). This paper is referred to as (I) in the present article. It contains references to previous work in the field. The notation used here is the same as that in (I). Reference to equations in (I) will be designated by the number of the equation preceded by the Roman numeral I.
[2] G. A. Alers and P. A. Fleury, Phys. Rev. **129**, 2425 (1963).
[3] T. Kjeldaas and T. D. Holstein, Phys. Rev. Letters **2**, 340 (1959).
[4] M. H. Cohen, M. J. Harrison, and W. A. Harrison, Phys. Rev. **117**, 937 (1960).

II. QUANTUM OSCILLATIONS

In this section we consider the oscillations in the velocity of sound in metals caused by the modification of the density of electron states by an applied magnetic field $\mathbf{B}_0$. The theory developed in (I) is still valid for the situation in which quantum effects occur, since the acceleration of the positive ions is certainly governed by a classical equation of motion. There are two changes to be made, however. The first is that Eq. I-(8) is not strictly valid. In fact, in the evaluation of the local[5] Fermi energy $\eta(\mathbf{r},t)$ one should use the density of energy states appropriate to an electron gas in the presence of the magnetic field. Let us designate this quantity, i.e., the number of energy levels per unit volume and per unit energy range at ϵ, by $g(\epsilon,B_0)$. Now, if the concentration of electrons at position $\mathbf{r}$ and time t is $n_0+n_1(\mathbf{r},t)$, the quantity $\eta=\eta(\mathbf{r},t)$ must be such that the following condition is satisfied:

$$n_0+n_1=\int_0^\infty g(\epsilon,B_0)f(\epsilon,\eta)d\epsilon. \quad (1)$$

Here $f(\epsilon,\eta)$ is identical to the Fermi function $f_0(\epsilon)$ except that we have substituted η instead of the equilibrium Fermi energy ζ. Since $n_1 \ll n_0$ we can make the approximation

$$f(\epsilon,\eta)=f_0(\epsilon)+(\eta-\zeta)(\partial f/\partial\eta)_{\eta=\zeta}$$
$$=f_0(\epsilon)-(\eta-\zeta)(\partial f_0/\partial\epsilon). \quad (2)$$

With the aid of the relation

$$n_0=\int_0^\infty g(\epsilon,B_0)f_0(\epsilon)d\epsilon, \quad (3)$$

[5] At this point we depart slightly from the notation in (I). The local Fermi energy is designated by $\eta(\mathbf{r},t)$, the Fermi energy in the absence of the acoustic wave but in the presence of $\mathbf{B}_0$ is called ζ, and $\zeta_0=(mv_0^2/2)$ is the Fermi energy in the absence of $\mathbf{B}_0$. However, for all practical purposes ζ and ζ_0 are quite close to each other. An expression for ζ in terms of ζ_0 is given in the Appendix.

we obtain

$$\eta(\mathbf{r},t)=\zeta+n_1\left[\int_0^\infty g(\epsilon,B_0)(-\partial f_0/\partial\epsilon)d\epsilon\right]^{-1}. \quad (4)$$

One can easily show, using the expressions for the energy eigenvalues of an electron in a magnetic field,[6] that

$$g(\epsilon,B_0)=(\hbar\omega_c/4\pi^2)(2m/\hbar^2)^{3/2} \times\sum_n\{\epsilon-(n+\tfrac{1}{2})\hbar\omega_c\}^{-1/2}, \quad (5)$$

where $\omega_c=-eB_0/mc$ is the cyclotron frequency and the sum over n extends from $n=0$ to the highest value of n for which the quantity under the radical is positive. It is possible to express $g(\epsilon,B_0)$, in a more convenient form, as the inverse Laplace transform of the partition function associated with a single electron in the presence of a magnetic field.[7] One obtains, when $\zeta_0/\hbar\omega_c\gg1$,

$$\int_0^\infty g(\epsilon,B_0)(-\partial f_0/\partial\epsilon)d\epsilon=3n_0/mv^2, \quad (6)$$

where

$$v^2=v_0^2\left\{1-\frac{\pi^2kT}{\zeta_0}\left(\frac{2\zeta_0}{\hbar\omega_c}\right)^{1/2}\sum_{\nu=1}^{\infty}(-1)^\nu \times\nu^{1/2}\frac{\cos[(2\pi\nu\zeta_0/\hbar\omega_c)-\frac{1}{4}\pi]}{\sinh(2\nu\pi^2kT/\hbar\omega_c)}-\frac{1}{12}\left(\frac{\hbar\omega_0}{2\zeta_0}\right)^2\right\}. \quad (7)$$

In this equation we have kept the leading oscillatory term and the first two monotonic contributions to the integral in Eq. (6). Clearly, this correction is only necessary for longitudinal waves since shear waves are not accompanied by changes in the density of the material. The second change is that instead of making use of the classical values for the conductivity tensor one must take their quantum mechanical expressions which may be found, for example, in two papers by Quinn and Rodriguez.[8,9] We have used the expression for the quantum mechanical conductivity tensor modified by the introduction of a phenomenological collision time in the same manner as it was done in Ref. 9. This assumption is, perhaps, a serious limitation in the present development, but without it little progress can be made. It seems hardly necessary to give here the details of the derivation. Nevertheless, for the purpose of reference, we give the expressions for the conductivity tensor in the Appendix.

First, we consider the case in which the direction of propagation of the acoustic wave is at right angles to the applied magnetic field. In this geometrical arrangement the frequencies ω and the polarizations $\boldsymbol{\xi}$ of the different acoustic modes are obtained, as a function of the wave vector $\mathbf{q}$, by solving the eigenvalue equation

$$\mathbf{A}\cdot\boldsymbol{\xi}=\omega^2\boldsymbol{\xi}. \quad (8)$$

If we take $\mathbf{q}$ parallel to the y axis and $\mathbf{B}_0$ parallel to the z axis of a Carteisan coordinate system, the nonvanishing components of $\mathbf{A}$ are [these formulas are quite quite similar to Eqs. I-(36)–I-(39) and are obtained in the same manner from I-(13) and I-(14)],

$$A_{xx}=\frac{C_t}{M}q^2+\frac{zmi\omega}{M\tau}\frac{(\sigma_0R_{xx}-1)(1-i\beta)}{1-i\beta\sigma_0R_{xx}}, \quad (9)$$

$$A_{xy}=-A_{yx}=\frac{zmi\omega}{M\tau}\left(\frac{(1-i\beta)\sigma_0R_{xy}}{1-i\beta\sigma_0R_{xx}}-\omega_c\tau\right), \quad (10)$$

$$A_{yy}=\frac{C_l}{M}q^2+\frac{zmq^2v^2}{3M(1+\omega^2\tau^2)}+\frac{zmi\omega}{M\tau} \times\left(\sigma_0R_{yy}-1-\frac{i\beta(\sigma_0R_{xy})^2}{1-i\beta\sigma_0R_{xx}}-\frac{(qv\tau)^2}{3(1+\omega^2\tau^2)}\right), \quad (11)$$

and

$$A_{zz}=\frac{C_t}{M}q^2+\frac{zmi\omega}{M\tau}\frac{(\sigma_0R_{zz}-1)(1-i\beta)}{1-i\beta\sigma_0R_{zz}}. \quad (12)$$

We notice that the only difference between these equations and the corresponding results in (I) is that, in A_{yy}, v_0 has been replaced by v. Expressions for the components of the tensor $\mathbf{R}$ are given in (I). Now, in the limit in which $\omega_c\tau(1+\omega^2\tau^2)^{-1/2}\gg1$, $\beta\ll1$, and $qv_0/\omega_c\ll1$, the approximate values of the nonvanishing components of $\mathbf{A}$ are

$$A_{xx}'=\frac{C_t}{M}q^2-\frac{zm\omega^2}{M}(1+\tfrac{1}{2}X\mu_1), \quad (13)$$

$$A_{xx}''=zm\omega X\mu_1/2M\tau, \quad (14)$$

$$A_{xy}'=-zm\omega\omega_c\beta X\mu_1/2M, \quad (15)$$

$$A_{xy}''=-\frac{zm\omega\omega_c}{M}\{X\mu_1+\beta\omega\tau(1-\tfrac{1}{2}X\mu_1)\}, \quad (16)$$

$$A_{yy}'=\frac{C_l}{M}q^2+\frac{zmq^2v^2}{3M(1+\omega^2\tau^2)}+\frac{4zm\omega^2X\mu_1(\omega_c\tau)^2}{M(1+\omega^2\tau^2)} -\frac{zm\omega^2}{M}+\frac{zm\omega}{M\tau}\beta(\omega_c\tau)^2(1-2X\mu_1), \quad (17)$$

$$A_{yy}''=\frac{zm\omega}{M\tau}\left[\frac{4X\mu_1(\omega_c\tau)^2}{1+\omega^2\tau^2}-\frac{(qv\tau)^2}{3(1+\omega^2\tau^2)}\right], \quad (18)$$

$$A_{zz}'=\frac{C_t}{M}q^2-\frac{zm\omega^2}{M}(1+4X\mu_z), \quad (19)$$

and

$$A_{zz}''=4zm\omega X\mu_z/M\tau. \quad (20)$$

[6] L. D. Landau, Z. Physik **64**, 629 (1930).
[7] A. H. Wilson, *The Theory of Metals* (Cambridge University Press, New York, 1958), 2nd ed., pp. 160–168.
[8] J. J. Quinn and S. Rodriguez, Phys. Rev. **128**, 2487 (1962).
[9] J. J. Quinn and S. Rodriguez, Phys. Rev. **128**, 2494 (1962).

The quantities X, μ_1, and μ_z are defined in the Appendix. It turns out that in the approximation considered here the terms in A_{xy} are negligible so that we can consider that the acoustic waves are purely longitudinal or purely transverse to a high degree of accuracy. In Eqs. (13)–(20), A_{ij}' and A_{ij}'' are the real and imaginary parts of A_{ij}, respectively $(i, j=x, y, z)$. For a wave polarized in the i direction we have

$$\omega^2=(\omega'+i\omega'')^2=A_{ii}\,, \tag{21}$$

and we can obtain both the velocity of sound $s=\omega'/q$ and the coefficient of attenuation $\gamma_a=2\omega''/s$. If $\omega\tau\ll 1$ the relative change in the sound velocity of a longitudinal acoustic wave propagating along the y axis ($\mathbf{B}_0$ is taken, as always, parallel to the z axis) is given approximately by

$$\frac{\Delta s}{s_l}=\frac{zm}{6Ms_l^2}(v^2-v_0^2)+\frac{B_0^2}{8\pi\rho s_l^2}\,. \tag{22}$$

The terms involving $X\mu_1$ can be seen to be negligible as compared with $zm(v^2-v_0^2)/6Ms_l^2$. It is necessary at this point to mention that Eq. (19) in Ref. 9 is not quite correct. In fact, in that paper, the effect of the magnetic field on the density of electron levels was disregarded. If we use Eq. (18) of the present work we obtain

$$\gamma_a(y,y)=\frac{zm}{M\tau s_l}\left[\frac{4X\mu_1(\omega_c\tau)^2}{1+\omega^2\tau^2}-\frac{(qv\tau)^2}{3(1+\omega^2\tau^2)}\right]. \tag{23}$$

The last term in Eq. (19) in Ref. 9 is, of course, still correct but in the present work we have neglected terms in β^2.

The velocity of shear waves also experiences quantum oscillations as a function of a magnetic field but the amplitude of these oscillations is smaller than that for longitudinal waves. Since the calculations are rather trivial once we have Eqs. (13)–(20) we shall not write down the results explicitly.

Alers and Swim[10] have recently measured the change in the velocity of sound in Au as a function of B_0 at liquid helium temperatures and using magnetic fields up to 10^5 G. They find a behavior that can be well represented by Eq. (22). If $2\pi^2kT>\hbar\omega_c$, we find

$$v^2-v_0^2\approx\frac{4\pi^2kT}{m}\left(\frac{2\zeta_0}{\hbar\omega_c}\right)^{1/2}\times\exp(-2\pi^2kT/\hbar\omega_c)\cos[(2\pi\zeta_0/\hbar\omega_c)-\tfrac{1}{4}\pi]\,. \tag{24}$$

This result can also be obtained using a purely thermodynamic argument. In fact, the velocity of longitudinal acoustic waves propagating in an isotropic solid (when $\omega\tau\ll 1$) is given by

$$s=(K\rho)^{-1/2}\,, \tag{25}$$

where K is the isothermal compressibility of the metal.[11] The bulk modulus K^{-1} is obtained from the Helmholtz free energy F using the thermodynamic equation

$$K^{-1}=V(\partial^2F/\partial V^2)_T=K_c^{-1}+K_e^{-1}. \tag{26}$$

Here K_c^{-1} is the contribution to the bulk modulus arising from the short-range forces among the ion cores and K_e^{-1} is the bulk modulus of the conduction electrons. We do not expect K_c to depend upon the applied magnetic field in a significant fashion. However, K_e does. In fact, Lifshitz and Kosevich[12] have shown that the free energy of the conduction electrons has an oscillatory contribution

$$F_{\text{osc}}=V\left(-\frac{eB_0}{\hbar c}\right)^{3/2}\frac{kT}{2\pi^2}\left\{\frac{2\pi}{|\partial^2S/\partial k_z^2|}\right\}^{1/2}\times\sum_{\nu=1}^{\infty}\frac{\cos[-(\nu\hbar cS_0/eB_0)-2\pi\nu\delta\mp\frac{1}{4}\pi]\cos(g\pi\nu m^*/2m)}{\nu^{3/2}\sinh[2\pi^2\nu kT/\hbar\omega_c]}\,. \tag{27}$$

This expression has been obtained using a theory of Onsager[13] for the description of the stationary states of a Bloch electron in the presence of a magnetic field. According to Onsager, the stationary states are such that the area $S(\epsilon,k_z)$ of the orbit in $\mathbf{k}$ space of an electron of energy ϵ and having a component k_z of its wave vector $\mathbf{k}$ along the direction of $\mathbf{B}_0$, must satisfy the condition

$$S(\epsilon,k_z)=-2\pi eB_0(n+\delta)/\hbar c\,. \tag{28}$$

Here n is a non-negative integer and δ a phase factor whose numerical value lies between zero and unity (for free electrons $\delta=\frac{1}{2}$). The quantity S_0 is the extremal cross-sectional area of the Fermi surface by planes perpendicular to $\mathbf{B}_0$. If more than one extremal cross section exists, then Eq. (27) contains a sum of terms each one arising from such an extremum. The upper sign of $\frac{1}{4}\pi$ is taken when the extremum is a maximum while the lower sign is used if it is a minimum. The symbol m^* stands for the cyclotron mass $m^*=(\hbar^2/2\pi)(\partial S/\partial\epsilon)_0$, where it is understood that the derivative is evaluated for an electron moving around the orbit for which S is an extremum. The cyclotron frequency ω_c is defined as $\omega_c=-eB_0/m^*c$. The quantity $\partial^2S/\partial k_z^2$ is to be evaluated on the Fermi surface and for that value of k_z for which S is an extremum. The term $\cos(g\pi\nu m^*/2m)$ owes its origin to the effect of the magnetic field on the orientation of the intrinsic magnetic moment of the electrons. Here g is the spectro-

[10] G. A. Alers and R. T. Swim, Phys. Rev. Letters **11**, 72 (1963).

[11] At low temperatures the adiabatic and isothermal compressibilities differ by a negligible amount. If $\omega\tau\ll 1$ it is easy to convince oneself that the electronic contribution to the velocity of sound arises from the isothermal compressibility of the conduction electrons.

[12] I. M. Lifshitz and A. M. Kosevich, Zh. Eksperim. i Teor. Fiz. **29**, 730 (1955) [translation: Soviet Phys.—JETP **2**, 636 (1956)].

[13] L. Onsager, Phil. Mag. **43**, 1006 (1952).

scopic splitting factor and m the free electron mass. There is also a nonoscillatory contribution to the free energy which contains a term equal to the free energy in the absence of a magnetic and another proportional to the square of B_0. Nevertheless, the latter is negligible for fields presently available in the laboratory. Using Eqs. (25), (26), and (27) we obtain, after making a few obvious approximations,

$$\frac{\Delta s}{s_l}=\frac{n_0^2}{\rho s_l^2[g(\zeta_0)]^2}\left(\frac{\hbar c}{eB_0}\right)^{1/2}\frac{kTm^{*2}}{\hbar^4}\left\{\frac{2\pi}{|\partial^2S/\partial k_z^2|}\right\}^{1/2}\sum_{\nu=1}^{\infty}\nu^{1/2}\frac{\cos[-(\nu\hbar cS_0/eB_0)-2\pi\nu\delta\mp\frac{1}{4}\pi]\cos(g\pi\nu m^*/2m)}{\sinh[2\pi^2\nu kT/\hbar\omega_c]}. \quad (29)$$

We have designated by $g(\epsilon)$ the density of energy states when $B_0=0$. If $2\pi^2kT>\hbar\omega_c$, then only the first term ($\nu=1$) in the sum on the left-hand side of Eq. (29) is important. If $\mathbf{B}_0$ points along a [111] direction of a Au crystal we expect a large oscillatory dependence of s with magnetic field. The reason for this is that in Au we have an extremal orbit[14] in this direction with $m^*=0.44m$ ("neck" orbit; for further details see Ref. 14 and other references therein). The period of the oscillations of s as a function of B_0^{-1} is

$$\Delta\left(\frac{1}{B_0}\right)=\frac{2\pi e}{\hbar cS_0}, \quad (30)$$

which is identical to what one expects in the de Haas–van Alphen effect.[15] Alers and Swim[10] have observed precisely this effect. Using Eq. (29) and $m^*=0.44m$ at $T=4°$K and $B_0=10^5$ G for Au, we find $\Delta s/s_l\simeq 20\times10^{-6}$ in reasonable agreement with the results of Alers and Swim. This calculation is rather crude, however. In fact, for lack of more precise information, we have used $|\partial^2S/\partial k_z^2|=2\pi$ and we have not considered the collision broadening of the Landau levels. Dingle[16] has shown that the collision broadening would give a further reduction of the first term in the sum of Eq. (29) by a factor $\exp(-2\pi/\omega_c\tau)$, where τ is the average relaxation time of the electron around the extremal orbit (Dingle's result is strictly valid only for a free electron gas).

We now consider briefly the case in which $\mathbf{q}$ is parallel to $\mathbf{B}_0$. The basic equations are again I-(25) and I-(26) with the exception that in I-(26) we must replace v_0 by v and $l=v_0\tau$ by $v\tau$. The results have been discussed in some detail in previous work[17] so that we shall not discuss this question again. It is interesting to remark however that Eq. (29) is valid here too for the propagation of a longitudinal acoustic wave parallel to the direction of $\mathbf{B}_0$. The quadratic increase $B_0^2/8\pi\rho s_l^2$ is not present in this geometrical arrangement. For the case of shear waves propagating in the direction of the applied magnetic field there does not seem to be appreciable quantum effects for fields of ordinary intensity and for acoustic frequencies that do not approach the conditions for which cyclotron resonance absorption occurs. This question has been discussed by Kjeldaas[18] using a classical model. In this geometry we expect a shear wave to experience a rotation of its plane of polarization as it progresses within the material. The angle of rotation is $-mc\omega^2/2\pi e\rho s_t^3$ per cm of path and per G of applied magnetic field. The foregoing result is applicable if $ql\ll1$ only. This angle of rotation is usually negligible for acoustic frequencies of the order of 10 Mc/sec but it may become appreciable at much higher frequencies.

Finally, we make two remarks about the result of Eq. (29). The first is that this equation is not strictly applicable to a material in which the Fermi surface possesses pieces in several bands. When this is the case, the compressions associated with the longitudinal wave may alter the energy discontinuities across the boundaries of the Brillouin zone in an appreciable way. This clearly gives rise to an additional change in the density of electron states which manifests itself in a corresponding change in the elastic constants. Price[19] has been able to accout for the experimental results of Mavroides *et al.*[20] in Bi considering that the change in the velocity of sound in this semimetal as a function of B_0 arises mainly from contributions to the elastic constants coming from transfer of electrons among the several valleys in the energy surfaces. The electrons are transferred among the several valleys because their relative position in energy is altered by the passage of the acoustic wave. The second remark is that there appears to be a contradiction between Eq. (29) which is obtained by a thermodynamic argument and Eq. (22) which is derived using the equation of motion for the acoustic wave. The difference between these two equations is the presence of the quantity $B_0^2/8\pi\rho s_l^2$ in the latter. This term is quite different from the one that originates in the nonoscillatory contribution to the free energy of the conduction electrons. In fact, the nonoscillatory part of the free energy is $F_0+F'=F_0-\frac{1}{2}\chi VB_0^2$, where χ is the constant part of the magnetic susceptibility (i.e., the sum of the Pauli and the Landau susceptibilities) and F_0 is the free energy when $B_0=0$. To establish the order

[14] D. Shoenberg, Phil. Trans. Roy. Soc. (London) **A255**, 85 (1962).

[15] See, for example, D. Shoenberg in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1957), Vol. 2, pp. 226–265.

[16] R. B. Dingle, Proc. Roy. Soc. (London) **A211**, 517 (1952).

[17] J. J. Quinn and S. Rodriguez, Phys. Rev. Letters **9**, 145 (1962); see also Ref. 8.

[18] T. Kjeldaas, Phys. Rev. **113**, 1473 (1959).

[19] P. J. Price (unpublished). The author is grateful to Dr. Price for making his work available to him.

[20] J. G. Mavroides, B. Lax, K. J. Button, and Y. Shapira, Phys. Rev. Letters **9**, 451 (1962).

of magnitude of this contribution to the velocity of sound we evaluate it for a particularly simple model. We assume a metal containing conduction electrons with spherical energy bands of effective mass m^*. The quantity F' is given by

$$F'=-\frac{V}{4\pi^2}\left(\frac{2m^*}{\hbar^2}\right)^{3/2}\zeta_0^{1/2}(\mu B_0)^2\left\{1-\frac{1}{3}\left(\frac{2m}{gm^*}\right)^2\right\}.$$

Here μ is the intrinsic magnetic moment of the electron and g the spectroscopic splitting factor. The relative change in the velocity of longitudinal acoustic waves turns out to be

$$\frac{\Delta s}{s_l}=\frac{z(\mu B_0)^2}{12\zeta_0 M s_l^2}\left\{1-\frac{1}{3}\left(\frac{2m}{gm^*}\right)^2\right\},$$

which is of the order of one part in 10^{10} for $B_0=10^5$ G and for a metal such as Au. Clearly, this amount is always negligible as compared with $B_0^2/8\pi\rho s_l^2$. This difficulty can be resolved by considering the frequency dependence of the last term in Eq. (22). At first sight it does not depend on the frequency at all. However, it is easy to convince oneself that if the dimensions of the sample are smaller than half of the wavelength of the acoustic disturbance, then $B_0^2/8\pi\rho s_l^2$ is absent from Eq. (22). The reason for this can be seen if we consider a longitudinal wave propagating along the y axis with $\mathbf{B}_0$ parallel to the z axis. Since, for ordinary ultrasonic frequencies, there is almost complete screening of the ions by the conduction electrons, the total current density in the y direction vanishes. However, the Lorentz forces on the ions and on the electrons due to the presence of $\mathbf{B}_0$ have different signs so that a Hall current density appears directed along the x axis and proportional to B_0. The magnetic field in turn interacts with the Hall current to give rise to a force per unit volume of material proportional to B_0^2. This is the origin of the quadratic dependence of Δs on B_0. However, if the dimension of the sample along the direction of propagation becomes less than half the acoustic wavelength, a depolarizing electric field can be established that reduces the magnitude of the Hall current. In fact, when we let ω approach zero, we expect $B_0^2/8\pi\rho s_l^2$ to disappear from the right-hand side of Eq. (22).

III. GEOMETRIC OSCILLATIONS

The geometric oscillations in the velocity of sound have been discussed to some extent in (I) so that our description here will be brief. These resonances manifest themselves when $\omega_c\tau\gg1$ and $ql\gg1$. The geometrical arrangement in which the acoustic wave propagates at right angles to the magnetic field $\mathbf{B}_0$ is of particular interest. We shall, in fact, limit our study to this geometry. If we assume $\omega_c\tau\gg1$, $\beta\ll1$, and that $\omega\tau$ does not appreciably exceed unity, the nonvanishing components of the tensor $\mathbf{A}$ are given by the approximate relations

$$A_{xx}=\frac{C_t}{M}q^2+\frac{zmi\omega}{M\tau}S_{xx}-\frac{zm\omega^2}{M}(1+S_{xx}),\tag{31}$$

$$A_{yy}=\frac{C_l}{M}q^2+\frac{zmq^2v_0^2}{3M}+\frac{zm\omega^2}{3M}\frac{(ql)^2}{1+\omega^2\tau^2}\times\left\{S_{yy}'+\frac{3\beta}{\omega\tau}(1+\omega^2\tau^2)S_{xy}^2\right\}+\frac{zmi\omega}{M\tau}(S_{yy}-1),\tag{32}$$

$$A_{xy}=-\frac{zmi\omega}{M}(\omega_c+qv_0S_{xy}),\tag{33}$$

and

$$A_{zz}=\frac{C_t}{M}q^2+\frac{zmi\omega}{M\tau}S_{zz}-\frac{zm\omega^2}{M}(1+S_{zz}).\tag{34}$$

In these equations, the expressions for S_{ij} are defined in (I) and

$$S_{yy}'=\frac{3(1+\omega^2\tau^2)}{q^2l^2}S_{yy}.\tag{35}$$

It was shown in (I) that it is possible to neglect the contributions arising from A_{xy}. A calculation of the sound velocity for shear waves polarized in the x and z directions and for longitudinal waves when $(3\beta/\omega\tau)\times(1+\omega^2\tau^2)\ll1$ was described in (I). The results were displayed in Figs. 1 and 2 of that work. However, if $qv_0/\omega_c\ll1$, $S_{xy}=-\omega_c/qv_0$ so that as $\omega_c\rightarrow\infty$ the graph displayed in Fig. 2 of (I) is not completely accurate for extremely large magnetic field. In fact, when $qv_0/\omega_c\ll1$, the term involving S_{xy} in Eq. (32), namely

$$\frac{zm\omega\beta(ql)^2S_{xy}^2}{M\tau}\approx\frac{B_0^2}{4\pi\rho}q^2.\tag{36}$$

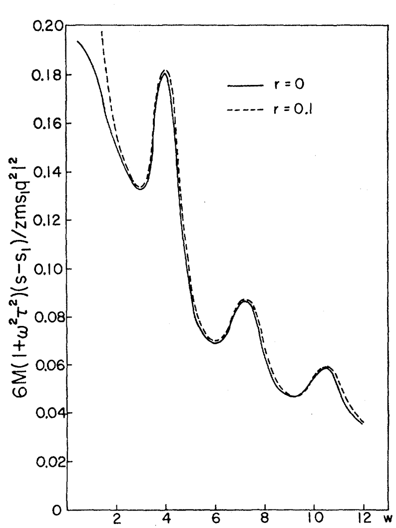

FIG. 1. Relative change in the velocity of longitudinal acoustic waves propagating at right angles to $\mathbf{B}_0$ for $r=0$ and $r=0.1$. The abscissa is the parameter $w=qv_0/\omega_c$.

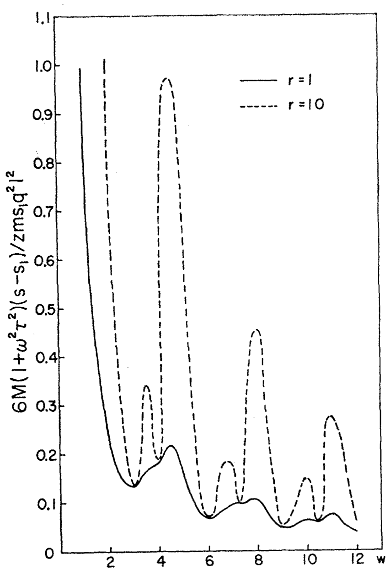

FIG. 2. Relative change in the velocity of longitudinal acoustic waves propagating at right angles to $\mathbf{B}_0$ as a function of w for $r=1$ and $r=10$.

This is once more the quadratic increase of the velocity of sound as a function of B_0. In Figs. 1 and 2 we give plots of the relative change in the velocity of longitudinal sound waves as a function of $w=qv_0/\omega_c$ for several values of the parameter

$$r=\frac{3\beta}{\omega\tau}(1+\omega^2\tau^2). \tag{37}$$

We do not extend our values to $w=0$ because in this region the term $B_0^2/4\pi\rho$ can exceed s_l^2 and then the velocity of sound would increase linearly with the intensity of applied magnetic field. The case in which $r=0$ is given as a limiting case. However, for sufficiently high magnetic fields, the term in r can become of importance.

It is interesting to notice that if $r\ll 1$ (see Fig. 1) the maxima and minima of $\Delta s/s_l$ occur at the same positions for which the ultrasonic attenuation has maxima and minima. In fact, the curve associated with $r=0$ in Fig. 1 is identical to the solid curve in Fig. 3 of Ref. 4. However, when $r>1$ we see from Fig. 2 of the present work that there are only minima of $\Delta s/s_l$ at the points w for which the graph for $r=0$ exhibited extrema. The reason for this behavior is clear. The maxima and minima of S_{yy} occur at the zeros of the function $g_0'(w)$. However, at these points S_{xy} vanishes and, if r is sufficiently large, S_{xy}^2 in Eq. (32) dominates the behavior of $\Delta s/s_l$ as a function of w.

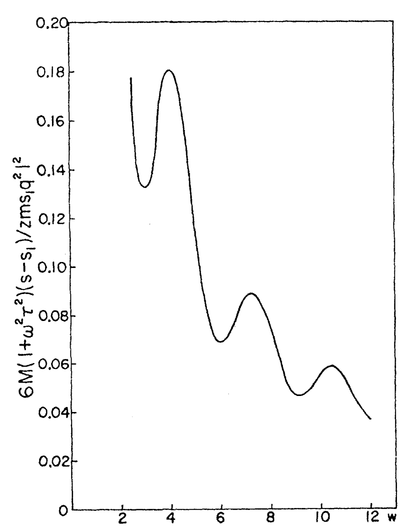

FIG. 3. Relative change in the velocity of longitudinal acoustic waves propagating at right angles to $\mathbf{B}_0$ as a function of w for $\beta\gg 1$.

Another case of interest occurs when $|\beta\sigma_0 R_{xx}|\gg 1$. For this condition to be satisfied over a wide range of values of w it is required that β itself be much larger than unity. This may occur at sufficiently high frequencies. If $\beta\gg 1$ and $\omega_c\tau(1+\omega^2\tau^2)^{-1/2}\gg 1$, the approximate value of A_{yy} is

$$A_{yy}=\frac{C_l}{M}q^2+\frac{zmq^2v_0^2}{3M}+\frac{zmi\omega}{M\tau}\left(S_{yy}-1+\frac{(ql)^2}{1+\omega^2\tau^2}\frac{S_{xy}^2}{1+S_{xx}}\right)$$
$$+\frac{zm\omega^2}{M}\left(S_{yy}+\frac{(ql)^2}{1+\omega^2\tau^2}\frac{S_{xy}^2}{1+S_{xx}}\right). \tag{38}$$

Then, the relative change in the velocity of sound is

$$\frac{\Delta s}{s_l}=\frac{zm(ql)^2}{6M(1+\omega^2\tau^2)}\left\{S_{yy}'+\frac{3S_{xy}^2}{1+S_{xx}}\right\}. \tag{39}$$

This result has been displayed in Fig. 3. The curve is, of course, identical to the dotted line in Fig. 3 of Ref. 4.

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APPENDIX

In this Appendix we give, for the purpose of reference, a number of results concerning the electrical conductivity tensor of a free electron gas in the presence of a dc magnetic field. Following Ref. 8 we designate the stationary states of an electron by the quantum numbers nk_yk_z and the corresponding energy eigenvalues by E_{nk_z}. The nonvanishing components of the electrical-conductivity tensor $\boldsymbol{\sigma}(\mathbf{q},\omega)$ for a magnetic field pointing along the z axis and $\mathbf{q}$ parallel to the y axis of a Cartesian coordinate system are

$$\sigma_{xx}=\frac{\sigma_0}{1+i\omega\tau}\left[1-\frac{4X}{N}\sum_{nk_yk_z}f_0(E_{nk_z})\times\sum_{\alpha=-n}^{\infty}\left(\frac{\partial f_{n+\alpha,n}}{\partial X}\right)^2\frac{\alpha}{\alpha^2+\gamma^2}\right], \tag{A1}$$

$$\sigma_{yy}=\frac{\sigma_0}{(\omega_c\tau)^2}(1+i\omega\tau)\frac{1}{NX}\sum_{nk_yk_z}f_0(E_{nk_z})\times\sum_{\alpha=-n}^{\infty}f_{n+\alpha,n}{}^2\frac{\alpha}{\alpha^2+\gamma^2},\tag{A2}$$

$$\sigma_{xy}=-\sigma_{yx}=-\frac{\omega_c\tau}{1+i\omega\tau}\frac{\partial}{\partial X}(X\sigma_{yy}),\tag{A3}$$

and

$$\sigma_{zz}=\frac{\sigma_0}{1+i\omega\tau}\left[1-\frac{4X}{N}\sum_{nk_yk_z}f_0(E_{nk_z})\left(\frac{k_z}{q}\right)^2\times\sum_{\alpha=-n}^{\infty}f_{n+\alpha,n}{}^2\frac{\alpha}{\alpha^2+\gamma^2}\right].\tag{A4}$$

Here

$$X=\hbar q^2/2m\omega_c,\tag{A5}$$

$$\gamma=(1+i\omega\tau)/\omega_c\tau,\tag{A6}$$

and

$$f_{n'n}(X)=(n!/n'!)^{1/2}X^{\frac{1}{2}(n'-n)}\times\exp(-\tfrac{1}{2}X)L_n{}^{(n'-n)}(X),\tag{A7}$$

if $n'\geq n$ and

$$f_{n'n}(X)=(-1)^{n'-n}f_{nn'}(X)\tag{A8}$$

if $n'<n$. The function $L_n{}^{(\alpha)}(X)$ is an associated Laguerre polynomial.[21]

For the study of the quantum effects which is developed in Sec. II of the text the following expansions of the components of $\boldsymbol{\sigma}$ have been used:

$$\sigma_{xx}=\sigma_0(1+i\omega\tau)^{-1}\left[4X\mu_1+\gamma^2-\frac{11}{2}X\gamma^2\mu_1-\gamma^4-6X^2\mu_2-\frac{1}{2}X^2\right],\tag{A9}$$

$$\sigma_{yy}=\sigma_0(\omega_c\tau)^{-2}(1+i\omega\tau)\left[1-\frac{3}{2}X\mu_1-\gamma^2+\frac{15}{8}X\gamma^2\mu_1+\gamma^4+\frac{5}{6}X^2\mu_2+\frac{7}{72}X^2\right],\tag{A10}$$

$$\sigma_{xy}=-\sigma_{yx}=-\sigma_0(\omega_c\tau)^{-1}\left[1-3X\mu_1-\gamma^2+\frac{15}{4}X\gamma^2\mu_1+\gamma^4+\frac{5}{2}X^2\mu_2+\frac{7}{24}X^2\right],\tag{A11}$$

and

$$\sigma_{zz}=\sigma_0(1+i\omega\tau)^{-1}(1-4X\mu_z).\tag{A12}$$

In these equations

$$\mu_\nu=N^{-1}\sum_{nk_yk_z}f_0(E_{nk_z})(n+\tfrac{1}{2})^\nu,\tag{A13}$$

and

$$\mu_z=\frac{X}{N}\sum_{nk_yk_z}\left(\frac{k_z}{q}\right)^2f_0(E_{nk_z}).\tag{A14}$$

The quantities μ_1 and μ_z have been used in Ref. 9, where $\mu_1=W/\hbar\omega_c$ and $\mu_z=W_z/\hbar\omega_c$. In the work of Sec. II we require the components of the resistivity tensor $\mathbf{R}$ rather than those of $\boldsymbol{\sigma}$. The nonvanishing components of $\mathbf{R}$ are

$$\sigma_0R_{xx}=(1+i\omega\tau)\left(1+\frac{1}{2}X\mu_1+\frac{47}{8}X\gamma^2\mu_1-2X^2\mu_1{}^2+\frac{11}{6}X^2\mu_2+\frac{1}{72}X^2\right),\tag{A15}$$

$$\sigma_0R_{yy}=(\omega_c\tau)^2(1+i\omega\tau)^{-1}\left(4X\mu_1+\gamma^2+\frac{1}{2}X\gamma^2\mu_1+8X^2\mu_1{}^2-6X^2\mu_2-\frac{1}{2}X^2\right),\tag{A16}$$

$$\sigma_0R_{xy}=(\omega_c\tau)\left(1-X\mu_1+\frac{25}{4}X\gamma^2\mu_1-5X^2\mu_1{}^2+\frac{7}{2}X^2\mu_2+\frac{5}{24}X^2\right),\tag{A17}$$

and

$$\sigma_0R_{zz}=(1+i\omega\tau)(1+4X\mu_z).\tag{A18}$$

We have kept σ_{zz} and R_{zz} to order $B_0{}^{-2}$ because this is as far as it is necessary to expand to obtain the most important oscillatory contributions to the real and imaginary parts of the tensor $\mathbf{A}$.

In the text we have defined a quantity v^2 [see Eq. (7)]. We must remark that v^2 is *not* equal $2\zeta/m$. In fact, the Fermi energy for conduction electrons of spherical effective mass m^* is

$$\zeta=\zeta_0\left[1-\left(\frac{\mu B_0}{2\zeta_0}\right)^2\left\{1-\frac{1}{3}\left(\frac{2m}{gm^*}\right)^2\right\}-\frac{\pi kT}{\zeta_0}\left(\frac{\hbar\omega_c}{2\zeta_0}\right)^{1/2}\times\sum_{\nu=1}^{\infty}\frac{(-1)^\nu\cos(g\pi\nu m^*/2m)\sin[(2\pi\nu\zeta_0/\hbar\omega_c)-\frac{1}{4}\pi]}{\nu^{1/2}\sinh(2\pi^2\nu kT/\hbar\omega_c)}\right].\tag{A19}$$

Here $\omega_c=-eB_0/m^*c$ as usual.

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[21] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, p. 188.